WA Exams Practice Paper A, 2016

Question/Answer Booklet

MATHEMATICS METHODS UNITS 3 AND 4

Section One: Calculator-free

SOLUTIONS

Student Number: In figures

In words ______________________________

Your name ______________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	99	65
			Total	151	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer Booklet.

3. You must be careful to confine your response to the specific question asked and to follow any instructions that are specified to a particular question.

4. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.
   - Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   - Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

5. **Show all your working clearly**. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula Sheet is **not** to be handed in with your Question/Answer Booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(4 marks)**

A function has a stationary point at (-2, 3) and is such that $f'(x) = 16 + kx^3$.

Determine the constant k and hence an equation for $f(x)$.

$$16 + k(-2)^3 = 0$$
$$8k = 16 \Rightarrow k = 2$$

$$f'(x) = 16 + 2x^3$$
$$f(x) = 16x + \frac{x^4}{2} + c$$

$$3 = 16(-2) + \frac{(-2)^4}{2} + c$$
$$3 = -32 + 8 + c$$
$$c = 27$$

$$f(x) = 16x + \frac{x^4}{2} + 27$$

Question 2 **(4 marks)**

Let $G(x) = \int_0^x \cos\left(2t + \frac{\pi}{6}\right) dt$.

(a) Determine $G'(x)$. (1 mark)

$$\frac{d}{dx}\int_0^x \cos\left(2t + \frac{\pi}{6}\right) dt = \cos\left(2x + \frac{\pi}{6}\right)$$

(b) Determine $G\left(\frac{\pi}{6}\right)$. (3 marks)

$$G\left(\frac{\pi}{6}\right) = \int_0^{\pi/6} \cos\left(2t + \frac{\pi}{6}\right) dt$$

$$= \left[\frac{1}{2}\sin\left(2t + \frac{\pi}{6}\right)\right]_0^{\pi/6}$$

$$= \frac{1}{2}\sin\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Question 3 **(6 marks)**

(a) Indicate, with reasons, which of the following could **not** represent a discrete probability distribution. (3 marks)

(i)

x	-1	0	1	2
$P(X=x)$	0.1	0.2	0.6	0.1

OK - is discrete pd.

(ii)

x	1	2	3	4
$P(X=x)$	0.4	0.5	-0.1	0.2

Could not, as individual probabilities must be between 0 and 1, and $P(X=3)$ is -0.1

(iii)

x	2	4	6	8
$P(X=x)$	0.3	0.2	0.3	0.4

Could not, as sum of probabilities must be 1, and in this case they sum to 1.2.

(b) The probability distribution for the number of broken eggs, X, in a carton of six sold at a supermarket is shown below.

x	0	1	2	3	4	5	6
$P(X=x)$	0.86	0.07	0.04	0.02	0.01	0.00	0.00

(i) Determine the probability that a randomly chosen egg carton contains at least four unbroken eggs. (1 mark)

$$P(\bar{X} \geq 4) = P(X \leq 2) = 0.97$$

(ii) Calculate $E(X)$. (2 marks)

$$\begin{aligned} E(X) &= 0\times 0.86 + 1\times 0.07 + 2\times 0.04 + 3\times 0.02 + 4\times 0.01 \\ &= 0.07 + 0.08 + 0.06 + 0.04 \\ &= 0.25 \end{aligned}$$

Question 4 **(8 marks)**

(a) Simplify $\log_2 32$. (2 marks)

$$\log_2 32 = \log_2 2^5$$
$$= 5\log_2 2$$
$$= 5$$

(b) Solve $2^{x-1} = 5$. (2 marks)

$$\log_2 5 = x - 1$$
$$x = \log_2 5 + 1$$

or

$$(x-1)\log 2 = \log 5$$
$$x\log 2 = \log 5 + \log 2$$
$$x = \frac{\log 5 + \log 2}{\log 2}$$

(c) If $x = \log_n 3$ and $y = \log_n 4$

(i) express $\log_n\left(\frac{4}{9}\right)$ in terms of x and/or y. (2 marks)

$$\log_n\left(\frac{4}{9}\right) = \log_n 4 - \log_n 3^2$$
$$= \log_n 4 - 2\log_n 3$$
$$= y - 2x$$

(ii) evaluate n^{2x}. (2 marks)

$$x = \log_n 3 \Rightarrow n^x = 3$$
$$\left(n^x\right)^2 = n^{2x} = 9$$

Question 5 **(9 marks)**

The continuous random variable X has probability density function $f(x)=\begin{cases}\dfrac{2x}{9} & 0\le x\le 3\\ 0 & \text{elsewhere}\end{cases}$.

(a) Determine $E(X)$. (2 marks)

$$E(X)=\int_0^3\left(x\times\frac{2x}{9}\right)dx$$
$$=\left[\frac{2x^3}{27}\right]_0^3=2$$

(b) The variance of X, $Var(X)$, is $\frac{1}{2}$.

(i) Determine $E(4X+3)$. (1 mark)

$$4\times2+3=11$$

(ii) Determine $Var(4X+3)$. (1 mark)

$$4^2\times\frac{1}{2}=8$$

(c) Determine the cumulative distribution function $F(x)$. (3 marks)

$$F(t)=\int_0^t\frac{2x}{9}dx$$
$$=\left[\frac{2x^2}{18}\right]_0^t=\frac{t^2}{9}$$
$$F(x)=\begin{cases}0 & x<0\\ \dfrac{x^2}{9} & 0\le x\le 3\\ 1 & x>3\end{cases}$$

(d) Calculate $P(1<X<2)$. (2 marks)

$$P(1<X<2)=F(2)-F(1)$$
$$=\frac{4}{9}-\frac{1}{9}=\frac{1}{3}$$

Question 6 **(7 marks)**

Let $f(x) = x^2 e^{2x-1}$.

(a) Determine the exact value of $f'(1)$. (3 marks)

$f'(x) = (2x)\left(e^{2x-1}\right) + \left(x^2\right)\left(2e^{2x-1}\right)$

$f'(1) = (2)(e) + (1)(2e) = 4e$

(b) Use the increments formula $\delta y \approx \dfrac{dy}{dx} \times \delta x$ with $x = 1$ to estimate $f(1.02)$. (4 marks)

When $x = 1$, $\dfrac{dy}{dx} = 4e$ and $\delta x = 0.02$.

$$\delta y \approx \frac{dy}{dx} \times \delta x$$
$$\delta y \approx 4e \times 0.02$$
$$\approx 0.08e$$

$$f(1) = e$$

$$f(1.02) \approx e + 0.08e$$
$$\approx 1.08e$$

Question 7 **(8 marks)**

A small storage tank, initially holding 20 L of water, is being filled so that the rate of change of volume of water in the tank t minutes after filling began is given by $\frac{dV}{dt} = \frac{10t}{t^2+1}$ for $t \geq 0$, where V is the volume of water in the tank, in litres.

(a) Use calculus to show that the tank is filling at the fastest rate after exactly 1 minute. (3 marks)

$$\frac{d^2V}{dt^2} = \frac{(10)(t^2+1) - (10t)(2t)}{\left(t^2+1\right)^2}$$

$$10t^2 + 10 - 20t^2 = 0$$

$$10 - 10t^2 = 0$$

$$t^2 = 1 \Rightarrow t = \cancel{-1},\ 1$$

(b) Determine an expression for V in terms of t. (3 marks)

$$V = \int \frac{10t}{t^2+1}\,dt$$

$$= 5\int \frac{2t}{t^2+1}\,dt$$

$$= 5\ln\left(t^2+1\right) + c$$

$$t = 0,\ V = 20 \Rightarrow c = 20$$

$$V = 5\ln\left(t^2+1\right) + 20$$

(c) The tank has a maximum capacity of 80 litres. Determine an exact expression for the time it will take to reach this capacity. (2 marks)

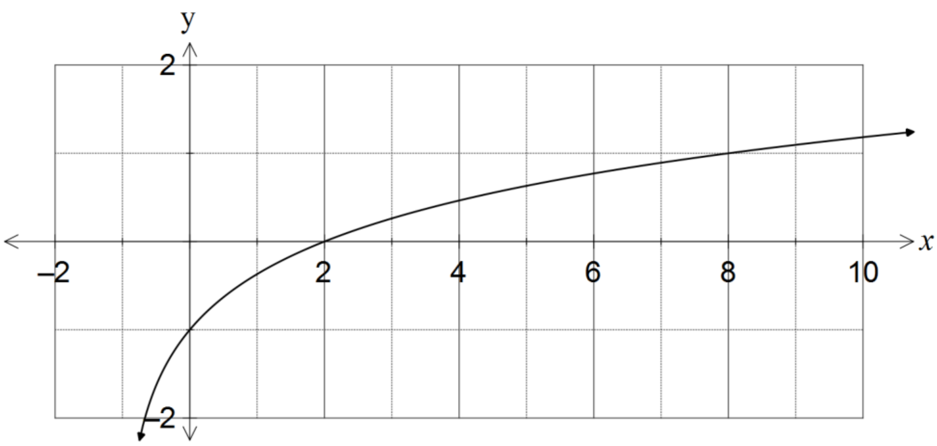
$$5\ln\left(t^2+1\right) + 20 = 80$$

$$\ln\left(t^2+1\right) = 12$$

$t = \sqrt{e^{12} - 1}$ minutes

Question 8 **(6 marks)**

The graph of $y = f(x)$, where $f(x) = \log_a(x+1) + b$, is shown below.



(a) Determine the values of a and b. (2 marks)

$$f(0) = \log_a(1) + b = -1 \;\Rightarrow\; b = -1$$

$$f(2) = \log_a(3) - 1 = 0 \;\Rightarrow\; a = 3$$

(b) If $\int_0^2 f(x)\,dx = p$ and $\int_2^4 f(x)\,dx = q$, determine, in terms of p and q

(i) $\int_0^4 f(x)\,dx$. (1 mark)

$$p + q$$

(ii) $\int_4^2 2f(x)\,dx$. (2 marks)

$$\int_4^2 2f(x)\,dx = -2\int_2^4 f(x)\,dx = -2q$$

(iii) the area between the x-axis and the curve $y = f(x)$ between $x = 0$ and $x = 4$. (1 mark)

$$-p + q$$

End of questions

Additional working space

Question number: _________